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Could only fermions be elementary?

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Abstract

In standard Poincare and anti de Sitter $SO(2, 3)$ invariant theories, antiparticles are related to negative energy solutions of covariant equations while independent positive energy unitary irreducible representations (UIRs) of the symmetry group are used for describing both a particle and its antiparticle. Such an approach cannot be applied in de Sitter $SO(1, 4)$ invariant theory. We argue that it would be more natural to require that (*) one UIR should describe a particle and its antiparticle simultaneously. This would automatically explain the existence of antiparticles and show that a particle and its antiparticle are different states of the same object. If (*) is adopted then among the above groups only the $SO(1, 4)$ one can be a candidate for constructing elementary particle theory. It is shown that UIRs of the $SO(1, 4)$ group can be interpreted in the framework of (*) and cannot be interpreted in the standard way. By quantizing such UIRs and requiring that the energy should be positive in the Poincare approximation, we conclude that (i) elementary particles can be only fermions. It is also shown that (ii) C invariance is not exact even in the free massive theory and (iii) elementary particles cannot be neutral. This gives a natural explanation of the fact that all observed neutral states are bosons.

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1. The statement of the problem

In standard quantum theory, the existence of antiparticles is explained as follows. Each elementary particle is described in two ways: (i) by using a unitary irreducible representation (UIR) of the Poincare (or anti de Sitter) group; (ii) by using a Poincare (or anti de Sitter) covariant equation. For each values of the mass and spin, there exist two UIRs—with positive and negative energies, respectively. At the same time, the corresponding covariant equation has solutions with both, positive and negative energies. As noted by Dirac (see, e.g., his Nobel lecture [1]), the existence of negative energy solutions represents a difficulty which should be resolved. In the standard approach, the solution is given in the framework of quantization

such that the creation and annihilation operators for the antiparticle have the usual meaning but they enter the quantum Lagrangian with the coefficients representing the negative energy solutions.

Such an approach has led to impressive success in describing various experimental data. However, as noted by Weinberg [2], 'this is our aim in physics, not just to describe nature, but to explain nature'. From this point of view, it seems unnatural that the covariant equation describes the particle and antiparticle simultaneously while UIRs for them are fully independent of each other. Moreover, UIRs with negative energies are not used at all.

The necessity to have negative energy solutions is related to the implementation of the idea that the creation or annihilation of an antiparticle can be treated, respectively as the annihilation or creation of the corresponding particle with the negative energy. However, since negative energies have no direct physical meaning in the standard theory, this idea is implemented implicitly rather than explicitly.

The above programme cannot be implemented if the de Sitter (dS) group $SO(1, 4)$ is chosen as the symmetry group. For example, it is well known that the dS Hamiltonian in UIRs has the spectrum in the interval $(-\infty, +\infty)$ (see, e.g., [3–7]). Note also that in contrast with the anti de Sitter (AdS) group $SO(2, 3)$, the dS one does not have a supersymmetric generalization. In view of modern approaches to local quantum field theory (LQFT) in curved spacetime, the dS group cannot be the symmetry group since, from the standpoint of any local observer, the vacuum has a finite temperature and admits particle destruction and creation (see, e.g., [8]). For this and other reasons, it was believed that the $SO(1, 4)$ group was not suitable for constructing elementary particle theory.

Although our approach considerably differs from that in [8] and references therein (see [9] for a detailed discussion) we come to the same conclusion (see section 3) that in the standard approach the dS group cannot be a symmetry group. However, the standard approach can be modified in such a way (see below) that theories with the dS symmetry become consistent. The main goal of the present paper is to investigate only the Poincare approximation where the standard physical intuition works. Some results of our investigations in the general case are mentioned in section 8.

It is well known that the group $SO(1, 4)$ is the symmetry group of the four-dimensional manifold in the five-dimensional space, defined by the equation

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2 = -R^2 \tag{1}$$

where a constant R has the dimension of length. The quantity R^2 is often written as $R^2 = 3/\Lambda$ where Λ is the cosmological constant. The nomenclature is such that $\Lambda < 0$ for the AdS symmetry while $\Lambda > 0$ for the dS one. The recent astronomical data show that, although Λ is very small, it is probably positive (see, e.g., [10]). For this reason the interest to dS theories has increased. Nevertheless, the existing difficulties have not been overcome (see, e.g., [11, 12]).

The fact that in the standard theory a particle and its antiparticle are treated independently poses a problem why they have equal masses, spins and lifetimes. The usual explanation (see, e.g., the standard textbooks [13]) is that this is a consequence of the CPT invariance. Therefore if it appears that the masses of a particle and its antiparticle were not equal, this would indicate the violation of the CPT invariance. In turn, as shown in well-known works [14], any local Poincare invariant quantum theory is automatically CPT invariant.

Such an explanation seems to be not quite convincing. Although at present there are no theories which explain the existing data better than the standard model based on LQFT, there is no guarantee that the ultimate quantum theory will be necessarily local. The modern theories aiming to unify all the known interactions (loop quantum gravity, noncommutative quantum

theory, string theory, etc) do not adopt the exact locality. Note also that the meaning of time operator is not quite clear [15] and, as it has been known already in 30th of the last century, when the quantum theory is combined with relativity, there is no operator which has all the required properties of the position operator (see, e.g., [16]). In particular, it is not possible to localize an object with the accuracy better than its Compton wavelength. For this and other reasons the quantity x in the Lagrangian density $L(x)$ is only a parameter which becomes the spacetime coordinate only in classical limit.

Consider a model example when isotopic invariance is exact (i.e. electromagnetic and weak interactions are absent). Then the proton and the neutron have equal masses and spins as a consequence of the fact that they belong to the same UIR of the isotopic group. In this example the proton and the neutron are simply different states of the same object—the nucleon, and the problem of why they have equal masses and spins has a natural explanation.

As shown in [17], in the quantum theory based on a Galois field, Galois field analogues of IRs of the AdS algebra $so(2, 3)$ have a property that a particle and its antiparticle are described by the same IR of the symmetry algebra. This automatically explains the existence of antiparticles and shows that a particle and its antiparticle represent different states of the same object. As argued in [9, 18], the description of the quantum theory in terms of Galois fields is more natural than the standard description based on the field of complex numbers. However, in the present paper we consider only the standard approach but with the following modification. Instead of saying that UIRs (by definition) describe elementary particles, we assume that:

Single-representation supposition. In standard quantum theory, any unitary irreducible representations of the symmetry group or algebra should describe a particle and its antiparticle simultaneously.

With such a requirement, among the Poincare, AdS and dS groups, only the latter can be a candidate for constructing the elementary particle theory. Therefore, we have to investigate whether UIRs of the dS group are compatible with the single-representation supposition. In section 2, we derive explicit expressions for representation generators in UIRs and their properties are discussed in section 4. In section 3, the Poincare approximation is discussed and it is shown that UIRs of the dS group cannot be interpreted in the standard way. In section 5, it is shown that the quantized version of UIR is indeed compatible with the single-representation supposition and in the Poincare approximation the energy can be positive definite only for fermions. As shown in section 6, the antiparticle generators satisfy the correct commutation relations but the particle and antiparticle generators are different. As a consequence (see section 7), elementary particles cannot be neutral, and even in the free massive dS invariant theory C invariance is only approximate. Finally, section 8 is discussion.

2. UIRs of the $SO(1, 4)$ group

As already noted, the de Sitter group $SO(1, 4)$ is the symmetry group of the four-dimensional manifold defined by equation (1). Elements of a map of the point $(0, 0, 0, 0, R)$ (or $(0, 0, 0, 0, -R)$) can be parametrized by the coordinates (x_0, x_1, x_2, x_3). If R is very large then such a map proceeds to Minkowski space and the action of the dS group on this map to the action of the Poincare group.

The representation generators of the $SO(1, 4)$ group M^{ab} $(a, b = 0, 1, 2, 3, 4, M^{ab} = -M^{ba})$ should satisfy the commutation relations

$$[M^{ab}, M^{cd}] = -\mathrm{i}(\eta^{ac}M^{bd} + \eta^{bd}M^{as} - \eta^{ad}M^{bc} - \eta^{bc}M^{ad}) \tag{2}$$

where η^{ab} is the diagonal metric tensor such that $\eta^{00} = -\eta^{11} = -\eta^{22} = -\eta^{33} = -\eta^{44} = 1$.

In conventional quantum theory elementary particles are described by UIRs of the symmetry group or IRs of its Lie algebra by self-adjoint operators in Hilbert spaces. Usually the latter also are called UIRs having in mind that the representation of the Lie algebra can be extended to the representation of the corresponding Lie group. We also will not discuss the difference between Hermitian and self-adjoint operators.

If one assumes that the role of the symmetry group is played by the Poincare group then the representations are described by ten generators—six generators of the Lorentz group and four components of the momentum operator. In the units $c = \hbar = 1$ the former are dimensionless while the latter has the dimension $(length)^{-1}$. If however, the symmetry group is $SO(1, 4)$ (or $SO(2, 3)$), then all the generators in the units $c = \hbar = 1$ are dimensionless.

The main goal of this section is to derive explicit expressions for the operators M^{ab} in the case of principal series of UIRs describing massive elementary particles. Although there exists a wide literature on UIRs of the $SO(1, 4)$ group (see, e.g., [19–22, 3–7]), we did not succeed in finding these expressions in the literature. For this reason, we will describe in detail the steps needed for derivation of equations (17) and (18).

The first complete mathematical classification of the UIRs has been given in [19], three well-known realizations of the UIRs have been first considered in [20] and their physical context has been first discussed in [21]. It is well known that for classification of UIRs we should, strictly speaking, consider not the group $SO(1, 4)$ itself but its universal covering group. The investigation carried out in [6, 19–22] has shown that this involves only replacement of the $SO(3)$ group by its universal covering group $SU(2)$. Since this procedure is well known, we will work with the $SO(1, 4)$ group itself and follow a very elegant presentation for physicists in terms of induced representations, given in the book [4] (see also [3, 22]).

The elements of the $SO(1, 4)$ group will be described in the block form

$$g = \begin{Vmatrix} g_0^0 & \mathbf{a}^{\mathrm{T}} & g_4^0 \\ \mathbf{b} & r & \mathbf{c} \\ g_0^4 & \mathbf{d}^{\mathrm{T}} & g_4^4 \end{Vmatrix} \tag{3}$$

where

$$\mathbf{a} = \begin{Vmatrix} a^1 \\ a^2 \\ a^3 \end{Vmatrix} \qquad \mathbf{b}^{\mathrm{T}} = \begin{Vmatrix} b_1 & b_2 & b_3 \end{Vmatrix} \qquad r \in SO(3) \tag{4}$$

(the superscript $^{\mathrm{T}}$ means a transposed vector).

UIRs of the $SO(1, 4)$ group are induced from UIRs of the subgroup H defined as follows [4]. Each element of H can be uniquely represented as a product of elements of the subgroups $SO(3)$, A and $\mathbf{T}$: $h = r\tau_A\mathbf{a_T}$ where

$$\tau_A = \begin{Vmatrix} \cosh(\tau) & 0 & \sinh(\tau) \\ 0 & 1 & 0 \\ \sinh(\tau) & 0 & \cosh(\tau) \end{Vmatrix} \qquad \mathbf{a_T} = \begin{Vmatrix} 1 + \mathbf{a}^2/2 & -\mathbf{a}^{\mathrm{T}} & \mathbf{a}^2/2 \\ -\mathbf{a} & 1 & -\mathbf{a} \\ -\mathbf{a}^2/2 & \mathbf{a}^{\mathrm{T}} & 1 - \mathbf{a}^2/2 \end{Vmatrix}. \tag{5}$$

The subgroup A is one-dimensional and the three-dimensional group $\mathbf{T}$ is the dS analogue of the conventional translation group (see, e.g., [4, 5]). We hope it should not cause misunderstandings when 1 is used in its usual meaning and when to denote the unit element of the $SO(3)$ group. It should also be clear when r is a true element of the $SO(3)$ group or belongs to the $SO(3)$ subgroup of the $SO(1, 4)$ group.

Let $r \to \Delta(r; \mathbf{s})$ be a UIR of the group $SO(3)$ with the spin $\mathbf{s}$ and $\tau_A \to \exp(\mathrm{i}\mu\tau)$ be a one-dimensional UIR of the group A, where μ is a real parameter. Then UIRs of the group H

used for inducing to the $SO(1,4)$ group, have the form

$$\Delta(r\tau_A \mathbf{a}_\mathbf{T};\mu,\mathbf{s}) = \exp(\mathrm{i}\mu\tau)\Delta(r;\mathbf{s}). \tag{6}$$

We will see below that μ has the meaning of the dS mass and therefore UIRs of the $SO(1,4)$ group are defined by the mass and spin, by analogy with UIRs in Poincare invariant theory.

Let $G = SO(1,4)$ and $X = G/H$ be the factor space (or coset space) of G over H. The notion of the factor space is well known (see, e.g., [3, 4, 23]). Each element $x \in X$ is a class containing the elements $x_G h$ where $h \in H$, and $x_G \in G$ is a representative of the class x. The choice of representatives is not unique since if x_G is a representative of the class $x \in G/H$ then $x_G h_0$, where h_0 is an arbitrary element from H, also is a representative of the same class. It is well known that X can be treated as a left G space. This means that if $x \in X$ then the action of the group G on X can be defined as follows: if $g \in G$ then gx is a class containing gx_G (it is easy to verify that such an action is correctly defined). Suppose that the choice of representatives is somehow fixed. Then $gx_G = (gx)_G(g,x)_H$ where $(g,x)_H$ is an element of H. This element is called a factor.

The explicit form of the generators M^{ab} depends on the choice of representatives in the space G/H. As explained in several papers devoted to UIRs of the $SO(1,4)$ group (see, e.g., [4]), to obtain the possible closest analogy between UIRs of the $SO(1,4)$ and Poincare groups, one should proceed as follows. Let $\mathbf{v}_L$ be a representative of the Lorentz group in the factor space $SO(1,3)/SO(3)$ (strictly speaking, we should consider $SL(2,C)/SU(2)$). This space can be represented as the well-known velocity hyperboloid with the Lorentz invariant measure

$$\mathrm{d}\rho(\mathbf{v}) = \mathrm{d}^3\mathbf{v}/v_0 \tag{7}$$

where $v_0 = (1+\mathbf{v}^2)^{1/2}$. Let $I \in SO(1,4)$ be a matrix which formally has the same form as the metric tensor η. One can show (see, e.g., [4] for details) that $X = G/H$ can be represented as a union of three spaces, X_+, X_- and X_0 such that X_+ contains classes $\mathbf{v}_L h$, X_- contains classes $\mathbf{v}_L I h$ and X_0 is of no interest for UIRs describing elementary particles since it has measure zero relative to the spaces X_+ and X_-.

As a consequence, the space of UIR of the $SO(1,4)$ group can be implemented as follows. If s is the spin of the particle under consideration, then we use $||\cdots||$ to denote the norm in the space of UIR of the group $SU(2)$ with the spin s. Then the space of UIR is the space of functions $\{f_1(\mathbf{v}), f_2(\mathbf{v})\}$ on two Lorentz hyperboloids with the range in the space of UIR of the group $SU(2)$ with the spin s and such that

$$\int [||f_1(\mathbf{v})||^2 + ||f_2(\mathbf{v})||^2]\,\mathrm{d}\rho(\mathbf{v}) < \infty. \tag{8}$$

We see that, in contrast with UIRs of the Poincare group (and AdS one), where UIRs are implemented on one Lorentz hyperboloid, UIRs of the dS group can be implemented only on two Lorentz hyperboloids, X_+ and X_-. Even this fact (which is well known) is a strong indication that UIRs of the dS group might have a natural interpretation in the framework of single-representation supposition (to the best of our knowledge, this possibility has not been considered in the literature).

In the case of Poincare and AdS groups, the positive energy UIRs are implemented on an analogue of X_+ and negative energy UIRs on an analogue of X_-. Since the Poincare and AdS groups do not contain elements transforming these spaces to one another, the positive and negative energy UIRs are fully independent. At the same time, the dS group contains such elements (e.g. I [4, 5]) and for this reason its UIRs cannot be implemented only on one hyperboloid.

A general construction of the operators M^{ab} is as follows. We first define right invariant measures on $G = SO(1,4)$ and H. It is well known (see, e.g., [23]) that for semisimple Lie

groups (which is the case for the dS group), the right invariant measure is simultaneously the left invariant one. At the same time, the right invariant measure $d_R(h)$ on H is not the left invariant one, but has the property $d_R(h_0h) = \Delta(h_0)d_R(h)$, where the number function $h \to \Delta(h)$ on H is called the module of the group H. It is easy to show [4] that

$$\Delta(r\tau_A\mathbf{a_T}) = \exp(-3\tau). \tag{9}$$

Let $\mathrm{d}\rho(x)$ be a measure on $X = G/H$ compatible with the measures on G and H [4] and let the representation space be implemented as the space of functions $\varphi(x)$ on X with the range in the space of UIR of the $SU(2)$ group such that

$$\int ||\varphi(x)||^2 \,\mathrm{d}\rho(x) < \infty. \tag{10}$$

Then the action of the representation operator $U(g)$ corresponding to $g \in G$ is defined as [4]

$$U(g)\varphi(x) = [\Delta((g^{-1}, x)_H)]^{-1/2}\Delta((g^{-1}, x)_H; \mu, \mathbf{s})^{-1}\varphi(g^{-1}x). \tag{11}$$

One can verify that this expression indeed defines a unitary representation. Its irreducibility can be proved in several ways (see, e.g., [4]).

As noted above, one can use the realization of the space X as the union of X_+ and X_- and then the representation space can be realized as in equation (8). Since we are interested in calculating the explicit form of representation generators, it is sufficient for this purpose to consider only elements of $g \in G$ in an infinitely small vicinity of the unit element of the dS group. In that case one can calculate the action of representation operators on functions having the carrier in X_+ and X_- separately. Namely, as follows from equation (11), for such $g \in G$, one has to find the decompositions

$$g^{-1}\mathbf{v}_L = \mathbf{v}'_L r'(\tau')_A(\mathbf{a}')_\mathbf{T} \tag{12}$$

and

$$g^{-1}\mathbf{v}_L I = \mathbf{v}''_L I r''(\tau'')_A(\mathbf{a}'')_\mathbf{T} \tag{13}$$

where $r', r'' \in SO(3)$. In this expressions, it is sufficient to consider only the elements of H belonging to an infinitely small vicinity of the unit element.

The problem of choosing representatives in the factor space $SO(1, 3)/SO(3)$ (or $SL(2.\mathrm{C})/SU(2)$) is well known in the standard theory. The most usual choice is such that $\mathbf{v}_L \in SO(1, 4)$ is represented by the matrix

$$\mathbf{v}_L = \begin{Vmatrix} v_0 & \mathbf{v}^\mathrm{T} & 0 \\ \mathbf{v} & 1 + \mathbf{v}\mathbf{v}^\mathrm{T}/(v_0 + 1) & 0 \\ 0 & 0 & 1 \end{Vmatrix}. \tag{14}$$

As follows from equations (6) and (11), there is no need to know the expressions for $(\mathbf{a}')_\mathbf{T}$ and $(\mathbf{a}'')_\mathbf{T}$ in equations (12) and (13). We can use the fact [4] that if e is the five-dimensional vector with the components $(e^0 = 1, 0, 0, 0, e^4 = -1)$ and $h = r\tau_A\mathbf{a_T}$, then $he = \exp(-\tau)e$ regardless of the elements $r \in SO(3)$ and $\mathbf{a_T}$. This makes it possible to easily calculate $(\mathbf{v}'_L, \mathbf{v}''_L, (\tau')_A, (\tau'')_A)$ in equations (12) and (13). Then one can calculate (r', r'') in these expressions by using the fact that the $SO(3)$ parts of the matrices $(\mathbf{v}'_L)^{-1}\mathbf{v}_L$ and $(\mathbf{v}''_L)^{-1}\mathbf{v}_L$ are equal to r' and r'', respectively.

The relation between the operators $U(g)$ and M^{ab} is as follows. Let L_{ab} be the basis elements of the Lie algebra of the dS group. These are the matrices with the elements

$$(L_{ab})^c_d = \delta^c_d\eta_{bd} - \delta^c_b\eta_{ad}. \tag{15}$$

They satisfy the commutation relations

$$[L_{ab}, L_{cd}] = \eta_{ac}L_{bd} - \eta_{bc}L_{ad} - \eta_{ad}L_{bc} + \eta_{bd}L_{ac}. \tag{16}$$

Comparing equations (2) and (16) it is easy to conclude that the M^{ab} should be the representation operators of $-\mathrm{i}L^{ab}$. Therefore if $g = 1 + \omega_{ab}L^{ab}$, where a sum over repeated indices is assumed and the ω_{ab} are such infinitely small parameters that $\omega_{ab} = -\omega_{ba}$ then $U(g) = 1 + \mathrm{i}\omega_{ab}M^{ab}$.

We are now in a position to write down the final expressions for the representation generators. The explicit calculation shows that the action of the generators on functions with the carrier in X_+ has the form

$$\begin{aligned}
&\mathbf{M}^{(+)} = l(\mathbf{v}) + \mathbf{s} \qquad \mathbf{N}^{(+)} = -\mathrm{i}v_0\frac{\partial}{\partial \mathbf{v}} + \frac{\mathbf{s}\times\mathbf{v}}{v_0+1} \\
&\mathbf{B}^{(+)} = \mu\mathbf{v} + \mathrm{i}\left[\frac{\partial}{\partial \mathbf{v}} + \mathbf{v}\left(\mathbf{v}\frac{\partial}{\partial \mathbf{v}}\right) + \frac{3}{2}\mathbf{v}\right] + \frac{\mathbf{s}\times\mathbf{v}}{v_0+1} \\
&M_{04}^{(+)} = \mu v_0 + \mathrm{i}v_0\left(\mathbf{v}\frac{\partial}{\partial \mathbf{v}} + \frac{3}{2}\right)
\end{aligned} \tag{17}$$

where $\mathbf{M} = \{M^{23}, M^{31}, M^{12}\}$, $\mathbf{N} = \{M^{01}, M^{02}, M^{03}\}$, $\mathbf{B} = -\{M^{14}, M^{24}, M^{34}\}$, $\mathbf{s}$ is the spin operator, and $\mathbf{l}(\mathbf{v}) = -\mathrm{i}\mathbf{v}\times\partial/\partial\mathbf{v}$. At the same time, the action of the generators on functions with the carrier in X_- is given by

$$\begin{aligned}
&\mathbf{M}^{(-)} = l(\mathbf{v}) + \mathbf{s} \qquad \mathbf{N}^{(-)} = -\mathrm{i}v_0\frac{\partial}{\partial \mathbf{v}} + \frac{\mathbf{s}\times\mathbf{v}}{v_0+1} \\
&\mathbf{B}^{(-)} = -\mu\mathbf{v} - \mathrm{i}\left[\frac{\partial}{\partial \mathbf{v}} + \mathbf{v}\left(\mathbf{v}\frac{\partial}{\partial \mathbf{v}}\right) + \frac{3}{2}\mathbf{v}\right] - \frac{\mathbf{s}\times\mathbf{v}}{v_0+1} \\
&M_{04}^{(-)} = -\mu v_0 - \mathrm{i}v_0\left(\mathbf{v}\frac{\partial}{\partial \mathbf{v}} + \frac{3}{2}\right).
\end{aligned} \tag{18}$$

3. Poincare approximation

Consider first the case when UIRs of the dS group can be contracted to UIRs of the Poincare group. A general notion of contraction has been developed in [24]. In our case it can be performed as follows. We assume that $\mu > 0$, denote $m = \mu/R$ and formally consider the limit $R \to \infty$ when $\mu \to \infty$ but μ/R is finite. We will consider the limits for equations (17) and (18) separately.

In the case of equation (17) we denote $\mathbf{P} = \mathbf{B}/R$ and $E = M_{04}/R$. Then in the above limit we obtain a standard representation of the Poincare group for a particle with the mass m such that $\mathbf{P} = m\mathbf{v}$ is the particle momentum and $E = mv_0$ is the particle energy. In that case the generators of the Lorentz group have the same form for the Poincare and dS groups. However, if the same procedure is applied to equation (18) then the quantity $\tilde{E} = M_{04}/R$ becomes negative and therefore the problem arises whether $\tilde{E}$ can be identified with the standard energy E.

The following important observation is in order. If we assume that the dS symmetry is more fundamental than the Poincare one then the limit $R \to \infty$ *should not* be actually taken since in this case the dS symmetry will be lost and the preceding consideration will become useless. However, in the framework of the dS invariance we can consider Poincare invariance as the approximate symmetry when R is very large but $R \neq \infty$. The situation is analogous to that when nonrelativistic theory is formally treated as a special case of relativistic one in the limit $c \to \infty$ and classical theory is treated as a special case of quantum one in the limit $\hbar \to 0$ but the limits are not actually taken. Summarizing these remarks, we prefer the term

'Poincare approximation' rather than 'Poincare limit'. *The term 'Poincare approximation' will always imply that R is very large but finite.*

In the standard interpretation of UIRs the following requirements should be satisfied:

- *Standard-interpretation requirements.* Each element of the full representation space represents a possible physical state for the given elementary particle. The representation describing a system of N free elementary particles is the tensor product of the corresponding single-particle representations.

  Recall that the generators of the tensor product are given by sums of the corresponding single-particle generators. For example, if $M_{04}^{(1)}$ is the operator M_{04} for particle 1 and $M_{04}^{(2)}$ is the operator M_{04} for particle 2 then the operator M_{04} for the free system $\{12\}$ is given by $M_{04}^{(12)} = M_{04}^{(1)} + M_{04}^{(2)}$. Here it is assumed that the action of the operator $M_{04}^{(j)}$ $(j = 1, 2)$ in the two-particle space is defined as follows. It acts according to equation (17) or (18) over its respective variables while over the variables of the other particle it acts as the identity operator.

  It is well known (see, e.g., [3–6]) that the dS group contains elements (e.g. I) such that the corresponding representation operator transforms eigenstates of M_{04} with the positive eigenvalues to the eigenstates of the same operator with the negative eigenvalues and vice versa. Therefore the problem arises whether the operator M_{04} in the single-particle UIRs can be treated as the dS analogue of the energy and what is the relation between M_{04} and the standard single-particle energy E. Let us stress that *if Poincare invariance is treated as a special case of the dS one then the quantity E cannot be defined independently and should somehow be expressed in terms of the operators* M^{ab}. One could try to remedy the standard interpretation as follows.
- Assume that in the Poincare approximation, the standard energy should be defined as

$$E = \pm M_{04}/R \tag{19}$$

  where the plus sign should be taken for the states with the carrier in X_+, and the minus sign for the states with the carrier in X_-. Then the energy will always be positive definite.
- One might say that the choice of the energy sign is only a matter of convention. Indeed, to measure the energy of a particle with the mass m one has to measure its momentum $\mathbf{p}$ and then the energy can be defined not only as $(m^2 + \mathbf{p}^2)^{1/2}$ but also as $-(m^2 + \mathbf{p}^2)^{1/2}$. In that case the standard energy in the Poincare approximation could be defined as

$$E = M_{04}/R \tag{20}$$

  regardless of whether the carrier of the given state is in X_+ or X_-.

It is easy to see that either of the above possibilities is incompatible with standard-interpretation requirements. Consider, for example, a system of two free particles in the Poincare approximation. Then with a high accuracy, the operators M_{04}/R and $\mathbf{B}/R$ can be chosen diagonal simultaneously.

Let us first assume that the energy should be treated according to equation (20). Then a system of two free particles with the equal masses can have the same quantum numbers as the vacuum (for example, if the first particle has the energy $E_0 = (m^2 + \mathbf{p}^2)^{1/2}$ and momentum $\mathbf{p}$ while the second one has the energy $-E_0$ and the momentum $-\mathbf{p}$) what obviously contradicts experiment. For this and other reasons, it is well known that in the Poincare invariant theory the particles should have the same energy sign. Analogously, if the single-particle energy is treated according to equation (19) and one requires that the two-body energy is the sum of single-particle energies then equation (19) for the two-body system will not be satisfied.

We conclude that UIRs of the dS algebra cannot be interpreted in the standard way since such an interpretation is physically meaningless even in the Poincare approximation. Although our approach considerably differs from LQFT in curved spacetime (see section 1), this conclusion is in agreement with that in [8] and references therein.

In the framework of single-representation supposition, one could try to interpret the operators (17) as those describing a particle while the operators (18) as those describing the corresponding antiparticle. This will be done in the subsequent sections and we will see that for quantized operators the energy can be interpreted as $E = M_{04}/R$.

It is clear that the above contraction procedure is valid only if $\mu \neq 0$. Therefore if we accept that Poincare invariant theory is a special case of the dS invariant one, the problem arises how to describe particles which in Poincaré invariant theory are strictly massless (and whether such particles can exist). Mensky has suggested [4] that massless particles should be described by UIRs corresponding to the additional series with $-i\mu = 1/2$. This problem requires further study.

4. Properties of representation generators

We now return to the general case when the quantity R is not necessarily large. Let us first compare equations (17) and (18). As follows from equation (2), if a set M_{ab} satisfies the correct commutation relations, the same is true for the set obtained from M_{ab} by changing the sign of those operators where $a = 4$ or $b = 4$ (the operator M_{44} is identical zero since $M_{ab} = -M_{ba}$). Therefore if one wants to verify that the operators (17) and (18) satisfy the conditions (2), it is sufficient to verify this either for (17) or (18).

It is obvious that the operators obtained from (17) or (18) by the transformation $\mu \to -\mu$ satisfy the conditions (2) if the original operators satisfy these conditions. Let us now apply the following transformation. First change the sign of μ and then change the sign of those operators M_{ab} where $a = 4$ or $b = 4$. Then we obtain a set of operators satisfying equation (2) if the original set satisfies equation (2). If such a transformation is applied to (17), we obtain the following set of operators:

$$\begin{aligned}
&\mathbf{M}' = l(\mathbf{v}) + \mathbf{s} \qquad \mathbf{N}' = -\mathrm{i}v_0\frac{\partial}{\partial \mathbf{v}} + \frac{\mathbf{s}\times\mathbf{v}}{v_0+1} \\
&\mathbf{B}' = \mu\mathbf{v} - \mathrm{i}\left[\frac{\partial}{\partial \mathbf{v}} + \mathbf{v}\left(\mathbf{v}\frac{\partial}{\partial \mathbf{v}}\right) + \frac{3}{2}\mathbf{v}\right] - \frac{\mathbf{s}\times\mathbf{v}}{v_0+1} \\
&M'_{04} = \mu v_0 - \mathrm{i}v_0\left(\mathbf{v}\frac{\partial}{\partial \mathbf{v}} + \frac{3}{2}\right).
\end{aligned} \tag{21}$$

By using equations (7) and (8), one can directly verify that the operators (17), (18) and (21) are Hermitian if the scalar product in the space of UIR is defined as follows. Since the functions $f_1(\mathbf{v})$ and $f_2(\mathbf{v})$ in equation (8) have the range in the space of UIR of the group $SU(2)$ with the spin s, we can replace them by the sets of functions $f_1(\mathbf{v}, j)$ and $f_2(\mathbf{v}, j)$, respectively, where $j = -s, -s+1, \ldots, s$. Moreover, we can combine these functions into one function $f(\mathbf{v}, j, \epsilon)$ where the variable ϵ can take only two values, say +1 or −1, for the components having the carrier in X_+ or X_-, respectively. If now $\varphi(\mathbf{v}, j, \epsilon)$ and $\psi(\mathbf{v}, j, \epsilon)$ are two elements of our Hilbert space, their scalar product is defined as

$$(\varphi, \psi) = \sum_{j,\epsilon}\int \varphi(\mathbf{v}, j, \epsilon)^*\psi(\mathbf{v}, j, \epsilon)\,\mathrm{d}\rho(\mathbf{v}) \tag{22}$$

where the superscript * applied to scalar functions means the usual complex conjugation.

At the same time, we use * to denote the operator adjoint to a given one. Namely, if A is the operator in our Hilbert space then A^* means the operator such that

$$(\varphi, A\psi) = (A^*\varphi, \psi) \tag{23}$$

for all such elements φ and ψ that the left-hand side of this expression is defined.

Even in the case of the operators (17), (18) and (21), we can formally treat them as integral operators with some kernels. Namely, if $A\varphi = \psi$, we can treat this relation as

$$\sum_{j',\epsilon'} \int A(\mathbf{v}, j, \epsilon; \mathbf{v}', j', \epsilon')\varphi(\mathbf{v}', j', \epsilon')\, \mathrm{d}\rho(\mathbf{v}') = \psi(\mathbf{v}, j, \epsilon) \tag{24}$$

where in the general case the kernel $A(\mathbf{v}, j, \epsilon; \mathbf{v}', j', \epsilon')$ of the operator A is a distribution.

As follows from equations (7), (23) and (24), if $B = A^*$ then the relation between the kernels of these operators is as follows:

$$B(\mathbf{v}, j, \epsilon; \mathbf{v}', j', \epsilon') = A(\mathbf{v}', j', \epsilon'; \mathbf{v}, j, \epsilon)^*. \tag{25}$$

In particular, if the operator A is Hermitian then

$$A(\mathbf{v}, j, \epsilon; \mathbf{v}', j', \epsilon')^* = A(\mathbf{v}', j', \epsilon'; \mathbf{v}, j, \epsilon). \tag{26}$$

As follows from equation (26), if the operator A is Hermitian, and its kernel is real then the kernel is symmetric, i.e.

$$A(\mathbf{v}, j, \epsilon; \mathbf{v}', j', \epsilon') = A(\mathbf{v}', j', \epsilon'; \mathbf{v}, j, \epsilon). \tag{27}$$

In particular, this property is satisfied for the operators μv_0 and $\mu\mathbf{v}$ in equations (17), (18) and (21). At the same time, the operators

$$l(\mathbf{v}) \qquad -\mathrm{i}v_0\frac{\partial}{\partial\mathbf{v}} \qquad -\mathrm{i}\left[\frac{\partial}{\partial\mathbf{v}} + \mathbf{v}\left(\mathbf{v}\frac{\partial}{\partial\mathbf{v}}\right) + \frac{3}{2}\mathbf{v}\right] \qquad -\mathrm{i}v_0\left(\mathbf{v}\frac{\partial}{\partial\mathbf{v}} + \frac{3}{2}\right) \tag{28}$$

which are present in equations (17), (18) and (21), are Hermitian but have imaginary kernels. Therefore, as follows from equation (26), their kernels are antisymmetric:

$$A(\mathbf{v}, j, \epsilon; \mathbf{v}', j', \epsilon') = -A(\mathbf{v}', j', \epsilon'; \mathbf{v}, j, \epsilon). \tag{29}$$

Note also that the operators considered in this paragraph do not depend on the spin and are present in equations (17), (18) and (21) for particles with arbitrary spins. At the same time, the spin operator is obviously different for particles with different spins. This question will be considered in section 6.

5. Quantization of UIRs

In standard approach to the quantum theory, the operators of physical quantities act in the Fock space of the given system. Suppose that the system consists of free particles and their antiparticles. Strictly speaking, in our approach it is not clear yet what should be treated as a particle or antiparticle. The considered UIRs of the dS group describe objects such that $(\mathbf{v}, j, \epsilon)$ is the full set of their quantum numbers. Therefore we can define the annihilation and creation operators $(a(\mathbf{v}, j, \epsilon), a(\mathbf{v}, j, \epsilon)^*)$ for these objects. If the operators satisfy the anticommutation relations then we require that

$$\{a(\mathbf{v}, j, \epsilon), a(\mathbf{v}', j', \epsilon')^*\} = \delta_{jj'}\delta_{\epsilon\epsilon'}v_0\delta^{(3)}(\mathbf{v} - \mathbf{v}') \tag{30}$$

while in the case of commutation relations

$$[a(\mathbf{v}, j, \epsilon), a(\mathbf{v}', j', \epsilon')^*] = \delta_{jj'}\delta_{\epsilon\epsilon'}v_0\delta^{(3)}(\mathbf{v} - \mathbf{v}'). \tag{31}$$

In the first case, any two a-operators or any two a^*-operators anticommute with each other while in the second case they commute with each other.

The problem of second quantization of representation operators can now be formulated as follows. Let $(A_1, A_2, \ldots, A_n)$ be representation generators describing UIR of the dS group. One should replace them by operators acting in the Fock space such that the commutation relations between their images in the Fock space are the same as for original operators (in other words, we should have a homomorphism of Lie algebras of operators acting in the space of UIR and in the Fock space). We can also require that our map should be compatible with the Hermitian conjugation in both spaces. It is easy to verify that a possible solution satisfying all the requirements is as follows. If the operator A in the space of UIR has the kernel $A(\mathbf{v}, j, \epsilon; \mathbf{v}', j', \epsilon')$ then the image of A in the Fock space is the operator

$$A_F = \sum_{j,\epsilon,j',\epsilon'} \iint A(\mathbf{v}, j, \epsilon; \mathbf{v}', j', \epsilon') a(\mathbf{v}, j, \epsilon)^* a(\mathbf{v}', j', \epsilon') \, \mathrm{d}\rho(\mathbf{v}) \, \mathrm{d}\rho(\mathbf{v}'). \tag{32}$$

The commutation relations in the Fock space will be preserved regardless of whether the (a, a^*) operators satisfy commutation or anticommutation relations.

We now require that in the Poincare approximation the energy should be positive definite. Recall that the operators (17), (18) act on their respective subspaces or in other words, they are diagonal in the quantum number ϵ.

Suppose that $\mu > 0$ and consider the quantized operator corresponding to the dS energy M_{04} in equation (17). In the Poincare approximation, $M_{04}^{(+)} = \mu v_0$ is fully analogous to the standard free energy and therefore, as follows from equation (32), its quantized form is

$$\left(M_{04}^{(+)}\right)_F = \mu \sum_j \int v_0 a(\mathbf{v}, j, 1)^* a(\mathbf{v}, j, 1) \, \mathrm{d}\rho(\mathbf{v}). \tag{33}$$

This expression is fully analogous to the standard quantized Hamiltonian if we assume that the vacuum state Φ_0 satisfies the requirement

$$a(\mathbf{v}, j, 1)\Phi_0 = 0 \qquad \forall\, \mathbf{v}, j. \tag{34}$$

In this case $a(\mathbf{v}, j, 1)$ has the meaning of the annihilation operator, $a(\mathbf{v}, j, 1)^*$ has the meaning of the creation operator, and $a(\mathbf{v}, j, 1)^*\Phi_0$ has the meaning of the one-particle state.

Consider now the operator $M_{04}^{(-)}$. In the Poincare approximation its quantized form is

$$\left(M_{04}^{(-)}\right)_F == -\mu \sum_j \int v_0 a(\mathbf{v}, j, -1)^* a(\mathbf{v}, j, -1) \, \mathrm{d}\rho(\mathbf{v}). \tag{35}$$

Therefore, if, by analogy with equation (34), one requires that

$$a(\mathbf{v}, j, -1)\Phi_0 = 0 \qquad \forall\, \mathbf{v}, j \tag{36}$$

then the operator $\left(M_{04}^{(-)}\right)_F$ will be negative definite, which is unacceptable.

Therefore the operators $a(\mathbf{v}, j, -1)$ and $a(\mathbf{v}, j, -1)^*$ are 'nonphysical': $a(\mathbf{v}, j, -1)$ is the operator of object's annihilation with the negative energy, and $a(\mathbf{v}, j, -1)^*$ is the operator of object's creation with the negative energy.

We will interpret the operator $\left(M_{04}^{(-)}\right)_F$ as that related to antiparticles. As already noted, in the standard approach, the annihilation and creation operators for antiparticles enter the quantum Lagrangian with the coefficients describing negative energy solutions of the corresponding covariant equation. This is an implicit implementation of the idea that the creation or annihilation of an antiparticle can be treated, respectively as the annihilation or creation of the corresponding particle with the negative energy. In our case this idea can be implemented explicitly.

Instead of the operators $a(\mathbf{v}, j, -1)$ and $a(\mathbf{v}, j, -1)^*$, we define new operators $b(\mathbf{v}, j)$ and $b(\mathbf{v}, j)^*$. If $b(\mathbf{v}, j)$ is treated as the 'physical' operator of antiparticle annihilation then, according to the above idea, it should be proportional to $a(\mathbf{v}, -j, -1)^*$. Analogously, if $b(\mathbf{v}, j)^*$ is the 'physical' operator of antiparticle creation, it should be proportional to

$a(\mathbf{v}, -j, -1)$. Therefore

$$b(\mathbf{v}, j) = a(\mathbf{v}, -j, -1)^*/\eta(j) \qquad b(\mathbf{v}, j)^* = a(\mathbf{v}, -j, -1)/\eta(j)^* \tag{37}$$

where $\eta(j)$ is a phase factor such that

$$|\eta(j)| = 1. \tag{38}$$

Since we treat $b(\mathbf{v}, j)$ as the annihilation operator and $b(\mathbf{v}, j)^*$ as the creation one, instead of equation (36) we should require that the vacuum condition should read

$$b(\mathbf{v}, j)\Phi_0 = 0 \qquad \forall\, \mathbf{v}, j \tag{39}$$

in the case of anticommutation relations

$$\{b(\mathbf{v}, j), b(\mathbf{v}', j')^*\} = \delta_{jj'} v_0 \delta^{(3)}(\mathbf{v} - \mathbf{v}') \tag{40}$$

and in the case of commutation relations

$$[b(\mathbf{v}, j), b(\mathbf{v}', j')^*] = \delta_{jj'} v_0 \delta^{(3)}(\mathbf{v} - \mathbf{v}'). \tag{41}$$

Consider first the case when the operators $a(\mathbf{v}, j, \epsilon)$ satisfy the anticommutation relations (30). By using equation (37) one can express the operators $a(\mathbf{v}, j, -1)$ in terms of the operators $b(\mathbf{v}, j)$. Then it follows from the condition (38) that the operators $b(\mathbf{v}, j)$ indeed satisfy equation (40).

Consider now the case when the operators $a(\mathbf{v}, j, \epsilon)$ satisfy the commutation relations (31). We can again use equation (37) to express the operators $a(\mathbf{v}, j, -1)$ in terms of the operators $b(\mathbf{v}, j)$. However, it now follows from the condition (38) that the operators $b(\mathbf{v}, j)$ do not satisfy equation (41) (they satisfy the equation obtained from equation (41) by changing the sign of the rhs).

These results show that only in the case of fermions our construction might be consistent. To see whether this is the case, we should express the operator (35) in terms of the operators $b(\mathbf{v}, j)$. By using equations (37) and (38), we can rewrite equation (35) as

$$\left(M_{04}^{(-)}\right)_F == -\mu \sum_j \int v_0 b(\mathbf{v}, j) b(\mathbf{v}, j)^* \mathrm{d}\rho(\mathbf{v}). \tag{42}$$

Now we have a situation fully analogous to that described in various textbooks (see, e.g., [13]) for quantizing the electron–positron field. It is well known that the only way to ensure the positive definiteness is to require that the operators $b(\mathbf{v}, j)$ and $b(\mathbf{v}, j)^*$ should satisfy the anticommutation relations (40). Then we can rewrite equation (42) as

$$\left(M_{04}^{(-)}\right)_F == \mu \sum_j \int v_0 b(\mathbf{v}, j)^* b(\mathbf{v}, j)\, \mathrm{d}\rho(\mathbf{v}) + C \tag{43}$$

where C is some indefinite constant. It can be eliminated by requiring that all quantized operators should be written in the normal form or by using another prescriptions. The existence of infinities in the standard approach is the well-known problem and we will not discuss it.

Our conclusion is as follows:

- *Statement 1:* The requirement that the Hamiltonian should be positive definite in the Poincare approximation, can be satisfied only for fermions.

6. Antiparticle sector

In the preceding section, we argued that the (b, b^*) operators are physical operators describing annihilation and creation of antiparticles. However, the proof of this statement has been given only in the Poincare approximation. To prove the statement in the general case we must show that the quantized operators (18) written in terms of the (b, b^*) operators satisfy the correct

commutation relations (2). We can use equations (37), (38) to express the operators $b(\mathbf{v}, j)$ in terms of $a(\mathbf{v}, j, -1)$, and, since we are now interested only in the case of anticommutation relations, we assume that equation (40) is satisfied.

Consider first the operators $-\mu v_0$ and $-\mu\mathbf{v}$ in equation (18). They are diagonal in the spin variable j. Assuming that all the quantized operators in terms of (b, b^*) are written in the normal form we easily conclude that

$$\begin{aligned}(-\mu v_0)_F &== \mu \sum_j \int v_0 b(\mathbf{v}, j)^* b(\mathbf{v}, j)\, \mathrm{d}\rho(\mathbf{v}) \\ (-\mu\mathbf{v})_F &== \mu \sum_j \int \mathbf{v} b(\mathbf{v}, j)^* b(\mathbf{v}, j)\, \mathrm{d}\rho(\mathbf{v}).\end{aligned} \tag{44}$$

Consider now the operators in equation (28). Let A be some of these operators and $A(\mathbf{v}, j, \epsilon; \mathbf{v}', j', \epsilon')$ be its kernel. Since A is diagonal in the spin variable j, it follows from equations (32), (37) and (38) that the action of A_F on functions with the carrier in X_- can be written as

$$A_F = \sum_j \int\int A(\mathbf{v}, j, -1; \mathbf{v}', j, -1) b(\mathbf{v}, j) b(\mathbf{v}', j)^*\, \mathrm{d}\rho(\mathbf{v})\, \mathrm{d}\rho(\mathbf{v}'). \tag{45}$$

As noted in section 4, the kernel of the operator A is antisymmetric. By using this fact and equation (40), we conclude that equation (45) can be rewritten as

$$A_F = \sum_j \int\int A(\mathbf{v}, j, -1; \mathbf{v}', j, -1) b(\mathbf{v}, j)^* b(\mathbf{v}', j)\, \mathrm{d}\rho(\mathbf{v})\, \mathrm{d}\rho(\mathbf{v}'). \tag{46}$$

In other words, the operator A_F has the same form in terms of (a, a^*) and (b, b^*) operators.

Finally, consider those operators in equation (18) which contain the lth component of the spin operator $\mathbf{s}$. Again, let A be some of these operators and $A(\mathbf{v}, j, -1; \mathbf{v}', j', -1)$ be the part of its kernel which is of interest for us. It is clear from equation (18), that in that case A is diagonal in $\mathbf{v}$, i.e. its kernel contains $v_0\delta(\mathbf{v} - \mathbf{v}')$. Therefore we can write the kernel in the form

$$A(\mathbf{v}, j, -1; \mathbf{v}', j', -1) = v_0\delta(\mathbf{v} - \mathbf{v}') f(\mathbf{v}) s^l_{jj'} \tag{47}$$

where $f(\mathbf{v})$ is a function of $\mathbf{v}$ and $s^l_{jj'}$ is the matrix element of s^l for the transition between the spin states j and j'. By using equation (32) we now obtain that the action of A_F on functions with the carrier in X_- is given by

$$A_F = \sum_{j,j'} \int f(\mathbf{v}) s^l_{jj'} a(\mathbf{v}, j, -1)^* a(\mathbf{v}, j', -1)\, \mathrm{d}\rho(\mathbf{v}). \tag{48}$$

As follows from equation (37), in terms of the (b, b^*) operators this expression reads

$$A_F = \sum_{j,j'} \int f(\mathbf{v}) s^l_{jj'} b(\mathbf{v}, -j) b(\mathbf{v}, -j')^* \eta(j)\eta(j')^*\, \mathrm{d}\rho(\mathbf{v}). \tag{49}$$

Since the trace of any spin operator equals zero, then by using equation (40), we can rewrite this expression as

$$A_F = -\sum_{j,j'} \int f(\mathbf{v}) s^l_{jj'} b(\mathbf{v}, -j')^* b(\mathbf{v}, -j) \eta(j)\eta(j')^*\, \mathrm{d}\rho(\mathbf{v}). \tag{50}$$

Consider first the case $l = 3$, i.e. A contains the z component of the spin operator. Since this component is diagonal in the spin index j, and j is the eigenvalue of the operator s^3, it

follows from equation (38) that

$$A_F = -\sum_j \int f(\mathbf{v}) j b(\mathbf{v}, -j)^* b(\mathbf{v}, -j)\, \mathrm{d}\rho(\mathbf{v}) = \sum_j \int f(\mathbf{v}) j b(\mathbf{v}, j)^* b(\mathbf{v}, j)\, \mathrm{d}\rho(\mathbf{v}). \quad (51)$$

We conclude that the operators containing the z component of the spin operator have the same form in terms of (a, a^*) and (b, b^*).

Consider now the operators containing s^l where $l = 1$ or $l = 2$. We choose $\eta(j)$ in the form $\eta(j) = (-1)^{(s-j)}$. Then, as follows from equation (50)

$$A_F = \sum_{j,j'} \int f(\mathbf{v})(s)^l_{jj'} b(\mathbf{v}, -j')^* b(\mathbf{v}, -j)\, \mathrm{d}\rho(\mathbf{v}) \quad (52)$$

since the operator s^l has nonzero matrix elements only for transitions with $j = j' \pm 1$. As follows from this expression, the operator A_F will have the same form in terms of (a, a^*) and (b, b^*) if

$$(s)^l_{j,j'} = (s)^l_{-j',-j}. \quad (53)$$

In the case $s = 1/2$ this relation can be easily verified directly. In the general case it can be proved by using the properties of $3j$ symbols (see, e.g., [25]). Therefore all the operators containing the components of $\mathbf{s}$ have the same form in terms of (a, a^*) and (b, b^*).

Our conclusion is as follows. If $A = \mu v_0$ or $A = \mu \mathbf{v}$ then the operator A_F has the same form in terms of (a, a^*) as $-A_F$ in terms of (b, b^*). At the same time, the other operators in equation (18) have the same form in terms of (a, a^*) and (b, b^*).

This result can be reformulated by saying that the quantized operators (18) can be obtained by quantizing operators (21) with (b, b^*) in place of (a, a^*). Since the operators (21) satisfy the required commutation relations (see the discussion in section 4), we conclude that for fermions the transformation defined by equations (37) and (38) is compatible with the commutation relations (2).

7. Discrete symmetries and nonexistence of neutral elementary particles

Let us now discuss the following question. For definiteness we assumed that $\mu > 0$. Will we get new UIRs if $\mu < 0$? In Poincare and AdS theories the choice of the mass sign implies simultaneously the choice of the energy sign and UIRs with the different mass signs are different. However, as we have seen in the preceding sections, in the dS case each UIR contains the states with both positive and negative eigenvalues of the operator M_{04}. A well-known result (see, e.g., [4] for details) is that UIRs characterized by μ and $-\mu$ are unitarily equivalent. For this reason, Mensky has proposed the following approach for distinguishing particles from antiparticles: they are described by the same UIRs but have different spacetime interpretation (see [4] for details). In this approach the UIRs are interpreted in the standard way (see section 3).

The fact that the same UIR of the dS group contains the states with both the positive and negative eigenvalues of the operator M_{04} is the reason of why the state of the object described by a UIR is characterized not only by the velocity $\mathbf{v}$ and the spin projection j but also by a new quantum number ϵ. In our approach one UIR describes a particle and its antiparticle simultaneously. For definiteness we assumed that $\epsilon = 1$ for particles and $\epsilon = -1$ for antiparticles but such a choice is obviously the matter of convention.

As it has been noted in section 1, in the standard theory the fact that a particle and its antiparticle have equal masses, spins and lifetimes is a consequence of CPT invariance. Let us discuss this problem in greater detail.

P invariance is described by a unitary transformation, which changes the signs of all three-dimensional momenta. In the standard theory one cannot define T invariance analogously since in that case the energy sign would change. There exist two well-known solutions of this difficulty: the Wigner formulation which involves antiunitary operators and the Schwinger formulation which involves transposed operators [13].

The comparison of equations (17) and (18) shows that the operators M_{ab} in these expressions not containing the subscript 4 are the same while those containing this subscript have different signs. If the coordinates x^ν ($\nu = 0, 1, 2, 3$) are inverted (i.e. one applies the PT transformation) and no antiunitary or transposed operators are used then the operators $M_{\nu 4}$ change their signs while the other operators remain unchanged. For these reasons one might think that the operators in equation (18) are obtained from ones in equation (17) by using the PT transformation. However, these equations have been obtained by considering only the elements of the $SO(1, 4)$ group belonging to its unity component, and no discrete transformations have been used. The matter is that the unitary representation operator corresponding to I necessarily changes the sign of the operator M_{04} in equation (17), i.e. it mimics the T transformation. Then if $\mathbf{v}$ is replaced by $-\mathbf{v}$ we obtain equation (18). To overcome the difficulty that the operator M_{04} in this equation is negative definite in the Poincare approximation, we relate equation (17) to particles, equation (18) to antiparticles and quantize these expressions in a proper way. In other words, our analogue of the PT transformation is accompanied by transition from particles to antiparticles, i.e. it is replaced by an analogue of the CPT transformation.

As noted by Mensky [4], dS invariant theory could be a basis for new approaches to the CPT theorem. We believe that our approach is in the spirit of Mensky's idea. An analogy between our approach and the standard CPT transformation is seen from the following observation. In the standard theory, the CPT transformation in Schwinger's formulation transforms the operators b to a^* [13] and for this reason one might think that equation (37) is the standard CPT transformation. However, in the standard theory the operators a and b refer to objects described by different UIRs while in our approach they refer to the same object. While in the standard theory the CPT transformation is a true transformation relating two sets of physical operators, equation (33) is not a transformation but *a definition* of the physical b operators in terms of unphysical a operators. In particular, $a^* = \eta b$ necessarily implies $a = \eta^* b^*$ and not $a = \eta b^*$ (i.e. there is no analogue of antiunitary transformation).

The operators $a(\mathbf{v}, j) == a(\mathbf{v}, j, 1)$ and $a(\mathbf{v}, j)^* == a(\mathbf{v}, j, 1)^*$ on one hand and $b(\mathbf{v}, j)$ and $b(\mathbf{v}, j)^*$ on the other satisfy the same commutation relations. As shown in section 5, the quantized representation generators for a particle are obtained from the operators (17) and (a, a^*). At the same time, the main result of section 6 is that the quantized representation generators for the corresponding antiparticle are obtained in the same way but with the operators (21) in place of the operators (17) and the operators (b, b^*) in place of the operators (a, a^*). Since the operators (17) and (21) are different (they coincide only in the limit $R \to \infty$), we conclude that different values of ϵ describe different sets of representation generators in the quantized form. Below we discuss this feature in detail since it has no analogue in the standard theory.

We first show that equations (17) and (21) are in agreement with the well-known results of general relativity (GR). For simplicity we consider the operators $M_{\nu 4}$ ($\nu = 0, 1, 2, 3$) in the nonrelativistic classical approximation. Denote $\mathbf{P} = \mathbf{B}/R$, $E = M_{04}/R$, $\mu = mR$, $\mathbf{p} = m\mathbf{v}$ and $\mathbf{r} = \mathrm{i}\partial/\partial\mathbf{p}$. Then as follows from equation (17)

$$\mathbf{P} = \mathbf{p} + m\mathbf{r}/R \qquad E = m + \mathbf{p}^2/2m + \mathbf{pr}/R \tag{54}$$

and, as follows from equations (21),

$$\mathbf{P} = \mathbf{p} - m\mathbf{r}/R \qquad E = m + \mathbf{p}^2/2m - \mathbf{pr}/R. \tag{55}$$

Therefore in the both cases the classical nonrelativistic Hamiltonian reads

$$E = m + \frac{\mathbf{P}^2}{2m} - \frac{m\mathbf{r}^2}{2R^2}. \tag{56}$$

Note that $\mathbf{r}$ is canonically conjugated with $\mathbf{p}$ by construction. In the approximation when R is large, the last term in the rhs of equation (56) is a small correction and $\mathbf{r}$ is also canonically conjugated with $\mathbf{P}$.

The well-known result of GR is that if the metric is stationary and differs slightly from the Minkowskian one then in the nonrelativistic approximation the curved spacetime can be effectively described by a gravitational potential $\varphi(\mathbf{r}) = (g_{00}(\mathbf{r}) - 1)/2c^2$ where g_{00} is the time–time component of the metric tensor. As follows from equation (1), in the approximation when R is large, the interval squared is given by

$$\mathrm{d}s^2 = \mathrm{d}x_\nu\,\mathrm{d}x^\nu - (x_\nu\,\mathrm{d}x^\nu/R)^2. \tag{57}$$

We now express x_0 in terms of a new variable t as $x_0 = t + t^3/6R^2 - t\mathbf{x}^2/2R^2$. Then

$$\mathrm{d}s^2 = \mathrm{d}t^2(1 - \mathbf{r}^2/R^2) - \mathrm{d}\mathbf{r}^2 - (\mathbf{r}\,\mathrm{d}\mathbf{r}/R)^2. \tag{58}$$

Therefore, the metric becomes stationary and $\varphi(\mathbf{r}) = -\mathbf{r}^2/2R^2$ in agreement with equation (56).

It is well known that in the dS space there exists antigravity: the force of repulsion between two particles is proportional to the distance between them. This easily follows from equation (56). We now show for illustrative purposes how this result can be obtained if $\mathbf{P}$ and E are expressed in terms of $\mathbf{p}$ and $\mathbf{r}$ as in equations (54) and (55).

Consider a system of two free particles described by the variables $\mathbf{p}_j$ and $\mathbf{r}_j$ $(j = 1, 2)$. Define the standard nonrelativistic variables

$$\begin{aligned} &\mathbf{P}_{12} = \mathbf{p}_1 + \mathbf{p}_2 \qquad \mathbf{q}_{12} = (m_2\mathbf{p}_1 - m_1\mathbf{p}_2)/(m_1 + m_2) \\ &\mathbf{R}_{12} = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/(m_1 + m_2) \qquad \mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2. \end{aligned} \tag{59}$$

Then if the particles are described by equation (54), the two-particle operators $\mathbf{P}$ and $\mathbf{E}$ in the nonrelativistic approximation are given by

$$\mathbf{P} = \mathbf{P}_{12} + M\mathbf{R}_{12}/R \qquad E = M + \mathbf{P}_{12}^2\big/2M + \mathbf{P}_{12}\mathbf{R}_{12}/R \tag{60}$$

where

$$M = M(\mathbf{q}_{12}, \mathbf{r}_{12}) = m_1 + m_2 + \mathbf{q}_{12}^2\big/2m_{12} + \mathbf{q}_{12}\mathbf{r}_{12}/R \tag{61}$$

and m_{12} is the reduced two-particle mass. Comparing equations (54) and (60), we conclude that M has the meaning of the two-body mass and therefore $M(\mathbf{q}_{12}, \mathbf{r}_{12})$ is the internal two-body Hamiltonian. As follows from equation (61), the classical equations of motion corresponding to this Hamiltonian imply that $\mathrm{d}^2\mathbf{r}_{12}/\mathrm{d}t^2 = \mathbf{r}_{12}/R^2$, i.e. the internal Hamiltonian (61) indeed describes the dS antigravity.

For a system of two antiparticles, the result is obviously the same since equation (55) can be formally obtained from equation (54) if R is replaced by $-R$. At the same time, in the case of a particle–antiparticle system a problem with the separation of external and internal variables arises. In any case the standard result can be obtained by using equation (56).

The above discussion shows that the set of operators given by equations (17) and (21) are compatible with GR. At the same time, these sets are not obviously the same. Recall that the (a, a^*) and (b, b^*) operators describe annihilation and creation of particles and antiparticle in the states with a given velocity $\mathbf{v}$. If we accept the main postulate of quantum theory that any self-adjoint operator represents a measurable physical quantity then the quantities defined by $\mathbf{p}$, $\mathbf{P}$ and E are measurable, at least in principle. Then we conclude that for particles

and antiparticles the operators $\mathbf{P}$ and E are expressed in terms of $\mathbf{p}$ differently. With our convention for the choice of the quantum number ϵ, equations (54) and (55) for both particles and antiparticles can be written as

$$\mathbf{P} = \mathbf{p} + \epsilon m\mathbf{r}/R \qquad E = m + \mathbf{p}^2/2m + \epsilon\mathbf{pr}/R. \tag{62}$$

Note also that in terms of $\mathbf{v}$ the Lorentz group generators in equations (17) and (21) are the same. Therefore if $\mathbf{v}$ is expressed in terms of $\mathbf{P}$ for particles and antiparticles then these expressions will become different.

In the standard theory the C transformation is defined as $a \leftrightarrow \eta_C b$ where η_C is the charge parity such that $|\eta_C|^2 = 1$. Since the sets of the operators (17) and (21) are different, the operators $\left(M_{ab}^{(+)}\right)_F$ do not transform into $\left(M_{ab}^{(-)}\right)_F$ under the C transformation and vice versa. Therefore our conclusion is as follows:

- *Statement 2:* Even in the free massive dS invariant theory the C invariance is not exact.

It is easy to show that the free massive dS theory is P invariant and therefore the CP invariance in this theory also is not exact.

In the literature (even the very serious one—see, e.g., Okun's book [26]) the following question is sometimes discussed. Suppose that a spaceship from an extraterrestrial civilization is approaching the Earth, and the aliens ask us whether the Earth is built of matter or antimatter. Fortunately, since the CP invariance in weak interactions is not exact, we can explain them that the Earth is built of matter, not antimatter. For example, we can tell them that the probability to find positrons in the K_L^0 meson decays is greater than the probability to find electrons. The above consideration shows that in the dS invariant theory this could be explained simpler.

In particle physics a particle is called neutral if it indistinguishable from its antiparticle. In particular, the C transformation transforms a neutral particle into itself. Suppose that the object described by a UIR is characterized by an additive quantum number q such that if Q is the operator of this number and Φ is a state such that $Q\Phi = Q_0\Phi$ then $a^*\Phi$ is a state where $Qa^*\Phi = (Q_0 + q)a^*\Phi$. Then, as follows from equation (37), a particle and its antiparticle automatically have opposite quantum numbers. However, if all the additive quantum numbers are equal to zero, we cannot use this criterion to distinguish a particle from its antiparticle. Nevertheless, it follows from the above discussion that in our approach

- *Statement 3:* Any elementary particle cannot be neutral.

8. Discussion

In the present paper, we have reformulated the standard approach to quantum theory as follows. Instead of requiring that each elementary particle is described by its own UIR of the symmetry algebra, we assume single-representation supposition (see section 1) that one UIR should describe a particle and its antiparticle simultaneously. In that case, among the Poincare, AdS and dS algebras, only the latter can be a candidate for constructing elementary particle theory.

We show in section 3 that UIRs of the dS algebra cannot be interpreted in the standard way since such an interpretation is physically meaningless even in the Poincare approximation. Although our approach considerably differs from LQFT in curved spacetime, this conclusion is in agreement with that in [8] and references therein.

The important ingredient of our construction is that nonphysical states are associated with antiparticles. In [12], problems with the dS theories have been discussed in the framework of the thermofield theory which was developed many years ago in many-body theory (see [12] for references). In this theory there also exist physical and nonphysical states but our

interpretation is essentially different. Nevertheless, it is interesting to note that similar ideas can work in approaches which considerably differ each other.

Although the single-representation supposition seems natural, it is a supposition. Therefore the problem arises whether it can be substantiated. As noted in section 1, the present investigation has been inspired by our results in quantum theory over a Galois field (GFQT) [17]. Here no analogue of the single-representation supposition is needed since any IR of the symmetry algebra automatically describes a particle and its antiparticle simultaneously.

The main idea of our approach is extremely transparent: *if there exists any criterion for separating physical and nonphysical states then only fermions can be elementary*. Indeed, the transition from nonphysical to physical states involves replacement of annihilation operators by creation ones and vice versa. Therefore if $a = \eta^* b^*$ then necessarily $a^* = \eta b$, $\{a, a^*\} = \eta\eta^*\{b, b^*\}$ and $[a, a^*] = -\eta\eta^*[b, b^*]$. It is easy to satisfy the condition $\eta\eta^* = 1$ but in the field of complex numbers it is impossible to satisfy the condition $\eta\eta^* = -1$.

The criterion used in the present work is that the energy should be positive definite in the Poincare approximation. Let us discuss this question in greater detail.

The dS group contains transformations which transform positive energy states to negative energy ones and vice versa. This does not necessarily represent a problem. As we have already mentioned, the standard treatment of UIRs does not apply in the dS case. Note that our results are based only on the properties of representation generators while the dS space has not been used at all (in section 7 we mentioned the dS space only for illustrative purposes to show that the results are compatible with GR when R is large). As noted in section 1, the classical spacetime cannot be fundamental in quantum theory (see, e.g., [9] for a detailed discussion; this conclusion is also in the spirit of Heisenberg's S-matrix programme). In any case, in the approximation when R is large, one UIR of the dS algebra asymptotically splits into well-known disjoint UIRs of the Poincare algebra with positive and negative energies, respectively. Therefore, single-representation supposition in this approximation is fully consistent.

In addition to statement 1 (see section 5) that only fermions can be elementary, it also follows from our consideration (see statements 2 and 3 in section 7) that

- Even in the free massive dS invariant theory the C invariance is not exact.
- Any elementary particle cannot be neutral.

Statements 1 and 3 give a natural explanation of the fact that neutral fermions have not been observed and all observed neutral states are bosons.

The famous Pauli spin-statistics theorem [27] in LQFT states that fermions necessarily have a half-integer spin while bosons—an integer spin. After the original Pauli proof, many authors investigated more general approaches to the spin-statistics theorem (see, e.g., [28] and references therein). Since in our approach only fermions can be elementary, the problem arises whether it is possible to prove that their spin is necessarily half-integer. In [17] we have considered Galois field analogues of IR of the $so(2, 3)$ algebra. It has been shown that in the GFQT, as a consequence of simple arithmetic considerations, the vacuum condition is consistent only for particles with a half-integer spin. At the same time, we did not succeed in proving that only fermions can be elementary. The matter is that in Galois fields the relation $\eta\eta^* = -1$ is not impossible.

As noted in section 2, the explicit description of the representation space and operators depends on the choice of representatives in a certain coset space. The choice adopted in the present paper is convenient in the Poincare approximation but when R is not asymptotically large there exist more natural choices (see, e.g., [4, 7, 9, 18]). Recall that in our approach one UIR describes an object; proceeding from our experience, we wish to separate the states of that object into those related to either a particle or its antiparticle. In other words, if H is the

representation space then we wish to find subspaces H_+ and H_- such that H is a direct sum of H_+ and H_-, H_+ represents all possible states for a particle and H_-—all possible states for its antiparticle. The results of [17, 29] show that in general, whatever separation criterion is used, representation generators have nonzero matrix elements for transitions between H_+ and H_-. Is this an indication that the very notion of particles and antiparticles has exact meaning only in the Poincare approximation? In particular, does the conservation of electric charge take place only in the Poincare approximation? These problems deserve further study.

The possibility that only fermions can be elementary is very appealing from the aesthetic point of view. Indeed, what was the reason for nature to create elementary fermions and bosons if the latter can be built of the former? A well-known historical analogy is that before the discovery of the Dirac equation, it was believed that nothing could be simpler than the Klein–Gordon equation for spinless particles. However, it has turned out that the spin $1/2$ particles are simpler since the covariant equation for them is of the first order, not the second one as the Klein–Gordon equation. A very interesting possibility (which has been probably considered first by Heisenberg) is that only spin $1/2$ particles could be elementary.

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